Examples of Riemann Integration from definition

Def : $\int_a^b f(x)dx = \lim_{\lambda(\Delta)\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \quad where\ \lambda(\Delta) = max\{\Delta x_i : i = 1,2,...,n\},\ \xi_i \in [x_{i-1}, x_i], \Delta$ is a partition of $[a,b]$.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

(I) Algebraic formulas

This involves formulas such as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 1 Evaluate $\int_1^4 x^2 dx$ from definition.

Consider the partition of [1, 4], $\Delta = \left\{1,\ 1+3\times\frac{1}{n}, 1+3\times\frac{2}{n}, ..., 1+3\times\frac{n}{n}\right\}$

$$\int_1^4 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+3\times\frac{i}{n}\right)^2 \left(\frac{3}{n}\right) = \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} \left(1+\frac{6i}{n}+\frac{9i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \left[\sum_{i=1}^{n} 1 + \frac{6}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \left[n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \left[n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[3 + 9\left(1+\frac{1}{n}\right) + \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= \underline{21}$$

EXERCISE 1 Evaluate $\int_0^1 x^3 dx$ from the first principles.

(II) Trigonometry formulas

EXAMPLE 2 Evaluate $\int_a^b \sin x dx$ from definition.

Consider the partition of [a, b], $\Delta = \left\{a,\ a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, ..., a+\frac{n(b-a)}{n} = b\right\}$

$$\int_a^b \sin x\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(a + \frac{i(b-a)}{n}\right) \times \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \frac{b-a}{2n \sin\left(\frac{b-a}{2n}\right)} \sum_{i=1}^{n} 2\sin\left(\frac{b-a}{2n}\right) \sin\left(a + \frac{i(b-a)}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{b-a}{2n}}{\sin\left(\frac{b-a}{2n}\right)} \sum_{i=1}^{n} \left\{\cos\left[a + \left(i-\frac{1}{2}\right)\frac{b-a}{n}\right] - \cos\left[a + \left(i+\frac{1}{2}\right)\frac{b-a}{n}\right]\right\}$$

$$= \lim_{n\to\infty} \frac{\frac{b-a}{2n}}{\sin\left(\frac{b-a}{2n}\right)} \left\{\cos\left[a + \frac{b-a}{2n}\right] - \cos\left[a + \left(n+\frac{1}{2}\right)\frac{b-a}{n}\right]\right\} \quad = \underline{\underline{\cos a - \cos b}}$$

EXERCISE 2 Evaluate $\int_0^{\frac{\pi}{2}} \cos x \, dx$ from the first principles.

(III) L'hospital Rule

More difficult problems employ the use of L'hospital rule or other properties on limit.

EXAMPLE 3 Evaluate $\int_0^1 a^x dx$ from definition.

Consider the partition of [0,1], $\Delta = \left\{ x_i = \frac{i}{n}, \ i = 0,1,2,\ldots,n \right\}$

$$\int_0^1 a^x dx = \lim_{n\to\infty} \sum_{i=1}^{n} a^{\frac{i}{n}} \times \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\sqrt[n]{a}\right)^i = \lim_{n\to\infty} \frac{1}{n} \frac{\sqrt[n]{a}\left[\left(\sqrt[n]{a}\right)^n - 1\right]}{\sqrt[n]{a} - 1}$$, applying the Geom. series formula.

$$= \lim_{n\to\infty} (a-1) \frac{1/n}{1 - a^{-\frac{1}{n}}}$$, by L'hospital Rule of $\frac{0}{0}$ type, treating n as a continuous variable.

$$= (a-1) \lim_{n\to\infty} \frac{-1/n^2}{-a^{-\frac{1}{n}} \times \frac{1}{n^2} \times \ln a}$$

$$= \frac{a-1}{\ln a} \lim_{n\to\infty} a^{-\frac{1}{n}} = \underline{\underline{\frac{a-1}{\ln a}}}$$

EXERCISE 3 Show that $\int_a^b e^x \, dx = e^b - e^a$ from the first principles.

(IV) Width of sub-intervals of the partition may not a constant

EXAMPLE 4 Evaluate $\int_2^4 x^2 dx$ from the first principles.

Consider the partition of [2,4], $\Delta = \{2, 2r, 2r^2, \ldots, 2r^i, \ldots, 4\}$

where $r = \left(\frac{4}{2}\right)^{1/n} = 2^{1/n}$, and so $r^n = 2$.

Since r > 1, $\lim_{n\to\infty} r = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1$

$\Delta x_i = 2r^{i-1} - 2r^i = 2r^i (1 - r^{-1})$, $\xi_i = 2r^{i-1}$

$\lambda = \max\{\Delta x_i\} = \Delta x_n = 4(1 - r^{-1})$

$$\lim_{n\to\infty} \lambda = \lim_{r\to 1} 4\left(1 - r^{-1}\right) = 4(1-1) = 0$$

$$\int_2^4 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2r^{i-1}\right)^2 \left(2r^i - 2r^{i-1}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} 8r^{3(i-1)}(r-1)$$

$$= \lim_{r\to 1} 8(r-1) \sum_{i=1}^{n} r^{3(i-1)} \overset{(G.P.)}{=} \lim_{r\to 1} 8(r-1) \frac{r^{3n} - 1}{r^3 - 1} = \lim_{r\to 1} \frac{8\left(2^3 - 1\right)}{r^2 + r + 1} = \underline{\underline{\frac{56}{3}}}$$

EXERCISE 4 Evaluate $\int_a^b x^k \, dx$, where $k \neq 1$ and $b > a > 0$, from the first principles.

Hint : Consider the partition of [a,b], $\Delta = \{a, ar, ar^2, \ldots, ar^i, \ldots, ar^n\}$, where $r = \sqrt[n]{\frac{b}{a}} > 1$

and $\xi_i = ar^{i-1}$.